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Stabilized finite-element method for the stationary Navier-Stokes equations

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Abstract. A stabilized finite-element method for the two-dimensional stationary incompressible Navier-Stokes equations is investigated in this work. A macroelement condition is introduced for constructing the local stabilized formulation of the stationary Navier-Stokes equations. By satisfying this condition, the stability of the $Q_1 - P_0$ quadrilateral element and the $P_1 - P_0$ triangular element are established. Moreover, the well-posedness and the optimal error estimate of the stabilized finite-element method for the stationary Navier-Stokes equations are obtained. Finally, some numerical tests to confirm the theoretical results of the stabilized finite-element method are provided.

Key words: error estimation, Navier-Stokes equations, stabilized finite element

1. Introduction

The development of appropriate finite-element methods is a key component in the search for efficient techniques for solving the incompressible Navier-Stokes problem. By use of a primitive-variable formulation, the importance of ensuring the compatibility of the approximations for the velocity and the pressure by satisfying the so-called inf-sup condition is widely understood. It is also well known that the simplest conforming low-order elements like the $P_1 - P_0$ (linear velocity, constant pressure) triangular element and $Q_1 - P_0$ (bilinear velocity, constant pressure) quadrilateral element are not stable.

During the last two decades there has been a rapid development in practical stabilization techniques for the $P_1 - P_0$ element and the $Q_1 - P_0$ element for solving the Stokes problem. For this purpose a local "macroelement condition" and some energy methods have been used. The use of such a macroelement condition as a means of verifying the (Babuška-Brezzi) inf-sup condition is a standard technique (see, for example, [1, Chapter II]); the basic idea was first introduced by Boland and Nicolaides [2], and independently by Stenberg [3]. The stabilized mixed finite-element approximation under consideration is based on a combination of the standard variational formulation of the Stokes problem and a bilinear form including a jump operator in the pressure. The discrete velocity u_h and the discrete pressure p_h are defined on finite-element subspaces X_h and M_h, related to conforming low-order elements like the $P_1 - P_0$ triangular element, or the $Q_1 - P_0$ quadrilateral element, which do not possess the properties required by the inf-sup condition. Recently, Kechkar and Silvester [4,5], Kay and Silvester [6], Norburn and Silvester [7] and Silvester and Wathen [8] pursued work which laid the foundations of the mathematical analysis and numerics of locally stabilized mixed finite-element methods for the Stokes problem.

The aim of this paper is to extend the work of Braess [9], Kechkar and Silvester [4,5], Kay and Silvester [6], Norburn and Silvester [7] and Pitkäranta and Saarinen [10] to the case of the stabilized finite-element method for solving the stationary Navier-Stokes equations. A macroelement condition is introduced for constructing the locally stabilized formulation of the stationary Navier-Stokes equations. By satisfying this condition, the stability of the $Q_1 - P_0$ quadrilateral element and the $P_1 - P_0$ triangular element are established. Moreover, we obtain the well-posedness and the optimal error estimate of the stabilized finite-element method for the stationary Navier-Stokes equations. Finally, we provide some numerical tests to confirm the efficiency of the stabilized finite-element method.

The outline of the paper is as follows. In the next section we introduce the mathematical setting of the stationary Navier-Stokes equations. In Section 3 we recall the notion of global and local stabilization of the $Q_1 - P_0$ quadrilateral element and the $P_1 - P_0$ triangular element based on the macroelement condition and prove the well-posedness of the stabilized finite-element method for the stationary Navier-Stokes equations. The optimal error estimate of the stabilized finite-element method is introduced in Section 4. Some numerical results are presented in Section 5, which show that the lowest-order locally stabilized finite-element method is efficient. We draw some conclusions in Section 6.

2. Functional setting of the Navier-Stokes problem

Let Ω be a bounded domain in R^2 assumed to have a Lipschitz continuous boundary $\partial\Omega$ and to satisfy a further condition stated in **(A1)** below. We consider the stationary Navier-Stokes equations

$$\begin{cases} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \quad \operatorname{div} u = 0 \quad x\in\Omega; \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where $u = (u_1(x), u_2(x))$ represents the velocity vector, $p = p(x)$ the pressure, $f = f(x)$ the prescribed body force, and $\nu > 0$ the viscosity.

For the mathematical setting of problem (2.1), we introduce the following Hilbert spaces

$$X = H_0^1(\Omega)^2, \quad Y = L^2(\Omega)^2, \quad M = L_0^2(\Omega) = \left\{ q\in L^2(\Omega) : \int_\Omega q\,\mathrm{d}x = 0 \right\}.$$

The spaces $L^2(\Omega)^m, m = 1, 2, 4$ are endowed with the L^2-scalar product and L^2-norm denoted by $(\cdot,\cdot)$ and $|\cdot|$. The spaces $H_0^1(\Omega)$ and X are equipped with the scalar product and norm

$$((u, v)) = (\nabla u, \nabla v), \quad \|u\| = (\nabla u, \nabla u)^{1/2}.$$

As mentioned above, we need a further assumption on Ω:

(A1) Assume that Ω is regular so that the unique solution $(v, q)\in(X, M)$ of the steady Stokes problem

$$-\Delta v + \nabla q = g, \quad \operatorname{div}\, v = 0 \text{ in } \Omega, \quad v|_{\partial\Omega} = 0,$$

for a prescribed $g\in Y$ exists and satisfies

$$\|v\|_2 + \|q\|_1 \le C_0|g|,$$

where $C_0 > 0$ is a constant depending on Ω and $\|\cdot\|_i$ denotes the usual norm of the Sobolev space $H^i(\Omega)$ or $H^i(\Omega)^2$ for $i = 1, 2$.

We also introduce the following Laplace operator

$$Au = -\Delta u, \quad \forall u \in D(A) = H^2(\Omega)^2 \cap X,$$

and the bilinear operator

$$B(u, v) = (u \cdot \nabla)v + \frac{1}{2}(\operatorname{div} u)v, \quad \forall u, v \in X,$$

Moreover, we define the continuous bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ on $X \times X$ and $X \times M$, respectively, by

$$a(u, v) = \nu((u, v)), \quad \forall u, v \in X, \quad d(v, q) = -(v, \nabla q) = (q, \operatorname{div} v), \quad \forall v \in X, \ q \in M,$$

and a generalized bilinear form on $(X, M) \times (X, M)$ by

$$\mathcal{B}((u, p); (v, q)) = a(u, v) - d(v, p) + d(u, q),$$

and a trilinear form on $X \times X \times X$ by

$$\begin{aligned} b(u, v, w) &= \langle B(u, v), w\rangle_{X' \times X} = ((u \cdot \nabla)v, w) + \frac{1}{2}((\operatorname{div} u)v, w) \\ &= \frac{1}{2}((u \cdot \nabla)v, w) - \frac{1}{2}((u \cdot \nabla)w, v), \quad \forall u, v, w \in X. \end{aligned}$$

We remark that the validity of assumption **(A1)** is known (see [11,12]) if $\partial\Omega$ is of C^2, or if Ω is a two-dimensional convex polygon. From assumption **(A1)**, it is easily shown [11] that

$$|v| \le \gamma_0 \|v\|, \quad \|v\| \le \gamma_0 |PAv|, \quad \|v\|_2 \le \gamma_1 |PAv|, \tag{2.2}$$

where P is the L^2-orthonormal projection of Y onto the space $\{v \in L^2(\Omega)^2 : \operatorname{div} v = 0 \text{ in } \Omega \text{ and } v\cdot n|_{\partial\Omega} = 0\}$, and $\gamma_0, \gamma_1, \ldots$ are positive constants depending only on Ω.

It is easy to verify that $\mathcal{B}$ and b satisfy the following important properties (see [1,4,6,11,13]):

$$\begin{cases} \nu\|u\|^2 = \mathcal{B}((u, p); (u, p)), \\ |\mathcal{B}((u, p); (v, q))| \le \gamma_2(\|u\| + |p|)(\|v\| + |q|), \\ \alpha_0(\|u\| + |p|) \le \sup\limits_{(v,q)\in(X,M)} \dfrac{\mathcal{B}((u, p); (v, q))}{\|v\| + |q|} \end{cases} \tag{2.3}$$

hold for all $(u, p),\ (v, q) \in (X, M)$ and constants $\gamma_2 > 0$ and $\alpha_0 > 0$,

$$b(u, v, w) = -b(u, w, v), \tag{2.4}$$

$$|b(u, v, w)| \le \frac{1}{2}c_0|u|^{1/2}\|u\|^{1/2}(\|v\| \ |w|^{1/2}\|w\|^{1/2} + |v|^{1/2}\|v\|^{1/2}\|w\|), \tag{2.5}$$

for all $u, v, w \in X$ and

$$|b(u, v, w)| + |b(v, u, w)| + |b(w, u, v)| \le c_1\|u\||Av||w|, \tag{2.6}$$

for all $u \in X, v \in D(A), w \in Y$, where $c_0, c_1, \ldots,$ are positive constants depending on the domain Ω.

Under the above notations, the variational formulation of the problem (2.1) reads as follows: find $(u, p) \in (X, M)$ such that for all $(v, q) \in (X, M)$:

$$\mathcal{B}((u, p); (v, q)) + b(u, u, v) = (f, v). \tag{2.7}$$

The following existence and uniqueness results are classical (see [1, Chapter IV] and [14, Chapter II]).

Theorem 2.1. *Assume that* ν *and* $f \in Y$ *satisfy the following uniqueness condition:*

$$1 - \frac{c_0 \gamma_0^2}{\nu^2} |f| > 0. \tag{2.8}$$

Then the problem (2.7) *admits a unique solution* $(u, p) \in (D(A) \cap X, H^1(\Omega) \cap M)$ *such that*

$$\|u\| \le \frac{\gamma_0}{\nu} |f|, \quad |Au| + \|p\|_1 \le c, |f|, \tag{2.9}$$

where γ_0 *and* c_0 *are defined in* (2.2) *and* (2.5)*, respectively.*

3. Stabilized finite-element approximation

In this section we apply the stabilized finite-element method developed for the Stokes equations to consider the numerical solution of the two-dimensional stationary incompressible Navier-Stokes equations (2.1). Let $h > 0$ be a real positive parameter. The finite-element subspace (X_h, M_h) of (X, M) is characterized by $\tau_h = \tau_h(\Omega)$, a partitioning of $\bar{\Omega}$ into triangles or quadrilaterals, assumed to be regular in the usual sense (see [1,4,6,15]), *i.e.*, for some σ and ω with $\sigma > 1$ and $0 < \omega < 1$,

$$h_K \le \sigma \rho_K \quad \forall K \in \tau_h, \tag{3.1}$$

$$|\cos \theta_{iK}| \le \omega, \quad i = 1, 2, 3, 4, \ \forall K \in \tau_h, \tag{3.2}$$

where h_K is the diameter of element K, ρ_K is the diameter of the inscribed circle of element K, and θ_{iK} are the angles of K in the case of a quadrilateral partitioning. The mesh parameter h is given by $h = \max\{h_K\}$, and the set of all interelement boundaries will be denoted by Γ_h.

The finite-element subspaces of interest in this paper are defined by setting

$$R_1(K) = \begin{cases} P_1(K) & \text{if } K \text{ is triangular,} \\ Q_1(K) & \text{if } K \text{ is quadrilateral,} \end{cases} \tag{3.3}$$

giving the continuous piecewise (bi)linear velocity subspace

$$X_h = \{v \in X : v_i|_K \in R_1(K), \quad i = 1, 2, \ \forall K \in \tau_h\},$$

and the piecewise constant pressure subspace

$$M_h = \{q \in M : q|_K \in P_0(K), \quad \forall K \in \tau_h\}.$$

Note that neither of these methods are stable in the standard Babuška-Brezzi sense; $P_1 - P_0$ triangle "locks" on regular grids (since there are more discrete incompressibility constraints than velocity degrees of freedom), and the $Q_1 - P_0$ quadrilateral is the most infamous example of an unstable mixed method, as elucidated by Sani *et al.* [16].

With the above choices of the velocity-pressure finite-element spaces $(X_h, M_h) \subset (X, M)$, a globally stabilized discrete formulation of the Navier-Stokes problem (2.7) can be defined as follows.

Definition 3.1. Globally stabilized formulation: *find* $(u_h, p_h) \in (X_h, M_h)$ *such that for all* $(v, q) \in (X_h, M_h)$:

$$\mathcal{B}_h((u_h, p_h); (v, q)) + b(u_h, u_h, v) = (f, v), \tag{3.4}$$

where

$$\mathcal{B}_h((u,p);(v,q)) = \mathcal{B}((u,p);(v,q)) + \beta\ \mathcal{C}_h(p,q), \quad \forall (u,p),\ (v,q) \in (X,M)$$
$$\mathcal{C}_h(p,q) = \sum_{e \in \Gamma_h} h_e \int_e [p]_e [q]_e \mathrm{d}s, \quad \forall p,q \in M,$$

and $[\cdot]_e$ *is the jump operator across* $e \in \Gamma_h$, *and* $\beta > 0$ *is the global stabilization parameter [17].*

In order to define a locally stabilized formulation of the Navier-Stokes problem, we introduce *a macroelement partitioning* Λ_h as follows: Given any subdivision τ_h, a macroelement partitioning Λ_h may be defined such that each macroelement $\mathcal{K}$ is a connected set of adjoining elements from τ_h. Every element K must lie in exactly one macroelement, which implies that macroelements do not overlap. For each $\mathcal{K}$, the set of interelement edges, which are strictly in the interior of $\mathcal{K}$, will be denoted by $\Gamma_{\mathcal{K}}$, and the length of an edge $e \in \Gamma_{\mathcal{K}}$ is denoted by h_e.

With these additional definitions a locally stabilized formulation of the Navier-Stokes problem (2.7) can be stated as follows.

Definition 3.2. Locally stabilized formulation: *find* $(u_h, p_h) \in (X_h, M_h)$, *such that for all* $(v,q) \in (X_h, M_h)$

$$\mathcal{B}_h((u_h,p_h);(v,q)) + b(u_h,u_h,v) = (f,v), \tag{3.5}$$

where

$$\mathcal{C}_h(p,q) = \sum_{\mathcal{K} \in \Lambda_h} \sum_{e \in \Gamma_{\mathcal{K}}} h_e \int_e [p]_e [q]_e \mathrm{d}s, \quad \forall p,q \in M,$$

$[\cdot]_e$ *is the jump operator across* $e \in \Gamma_{\mathcal{K}}$ *and* $\beta > 0$ *is the local stabilization parameter.*

A general framework for analyzing the locally stabilized formulation (3.5) can be developed using the notion of equivalence class of macroelements. As in Stenberg [3], each equivalence class, denoted by $\mathcal{E}_{\hat{\mathcal{K}}}$, contains macroelements which are topologically equivalent to a reference macroelement $\hat{\mathcal{K}}$. To illustrate the idea, two practical examples of locally stabilized mixed approximations are given below.

Example 3.1. The first example is the standard $Q_1 - P_0$ approximation pair. A locally stabilized formulation (3.5) can be constructed in this case, if τ_h is such that the elements K can be grouped into 2×2 macroelements $\mathcal{K} = \{K_1,\ K_2,\ K_3,\ K_4\}$, with the reference macroelement

$$\hat{\mathcal{K}} = \{\hat{K}_1,\ \hat{K}_2,\ \hat{K}_3,\ \hat{K}_4\},$$

and arbitrary $\mathcal{K} \in \Lambda_h$ as illustrated in Figure 1.

An obvious way of constructing such a partitioning in practice is to form the grid τ_h by uniformly refining a coarse grid Λ_h, for example, by joining the mid-edge points.

Example 3.2. The triangular $P_1 - P_0$ approximation pair can similarly be established if the partitioning τ_h is constructed such that the elements can be grouped into disjoint macroelements, all consisting of four elements as illustrated in Figure 2.

For the above finite-element spaces X_h and M_h, it is well-known that the following approximation properties

$$|v - I_h v| + h\|v - I_h v\| \le c_3 h^2 |Av|, \quad \forall v \in D(A), \tag{3.6}$$
$$|q - J_h q| \le c_3 h \|q\|_1, \quad \forall q \in H^1(\Omega) \cap M, \tag{3.7}$$

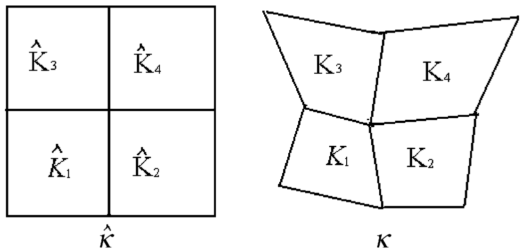

Figure 1. Reference and arbitrary ($Q_1 - P_0$) macroelements.

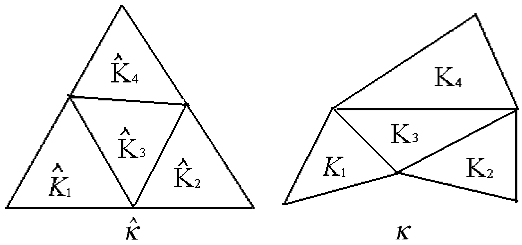

Figure 2. Reference and arbitrary ($P_1 - P_0$) macroelements.

and the inverse inequality

$$\|v_h\| \le c_3 h^{-1}|v_h|, \quad \forall v_h \in X_h, \tag{3.8}$$

hold (see [13], [15, Chapter III] and [1, Chapter III]), where $I_h : D(A) \to X_h$ is the interpolation operator and $J_h : H^1(\Omega) \cap M \to M_h$ is the L^2-orthogonal projection.

The following stability results of these mixed methods for the macroelement partitioning defined above were formally established by Kay and Silvester [6] and Kechkar and Silvester [4].

Theorem 3.3. *Given a stabilization parameter $\beta \ge \beta_0 > 0$, suppose that every macroelement $\mathcal{K} \in \Lambda_h$ belongs to one of the equivalence classes $\mathcal{E}_{\hat{\mathcal{K}}}$, and that the following macroelement connectivity condition is valid: for any two neighboring macroelements $\mathcal{K}_1$ and $\mathcal{K}_2$ with $\int_{\mathcal{K}_1 \cap \mathcal{K}_2} \mathrm{d}s \ne 0$ there exists $v \in X_h$ such that*

$$\operatorname{supp} v \subset \mathcal{K}_1 \cup \mathcal{K}_2 \text{ and } \int_{\mathcal{K}_1 \cap \mathcal{K}_2} v \cdot n \,\mathrm{d}s \ne 0. \tag{3.9}$$

Then,

$$|\mathcal{B}_h((u,p);(v,q))\| \le \gamma_3(\|u\| + |p|)(\|v\| + |q|), \quad \forall (u,p),(v,q) \in (X, M), \tag{3.10}$$

$$\alpha(\|u_h\| + |p_h|) \le \sup_{(v,q)\in(X_h,M_h)} \frac{\mathcal{B}_h((u_h,p_h);(v,q))}{\|v\| + |q|}, \quad \forall (u_h,p_h) \in (X_h, M_h), \tag{3.11}$$

$$|\mathcal{C}_h(p - J_h p, q_h)| \le c_4 h \|p\|_1 |q_h|, \quad \mathcal{C}_h(p, q_h) = 0, \quad \forall p \in H^1(\Omega) \cap M, q_h \in M_h, \tag{3.12}$$

where $\alpha > 0, \gamma_3 > 0$ are two constants independent of h and β, and β_0 is any fixed positive constant and n is the outnormal vector.

Throughout the article we shall assume that $\beta \ge \beta_0$.

Theorem 3.4. *Under the assumptions of Theorem* 2.1 *and Theorem* 3.3, *the problem* (3.5) *admits a unique solution $(u_h, p_h) \in (X_h, M_h)$ satisfying*

$$\|u_h\| \le \frac{\gamma_0}{\nu}|f|, \quad |p_h| \le \alpha^{-1}\left(c_0 \nu^{-2} \gamma_0^3 |f|^2 + \gamma_0 |f|\right). \tag{3.13}$$

Proof. Let the Hilbert space $H_h = (X_h, M_h)$ be supplied with the scalar product and norm:

$$((v,q);(w,r))_{H_h} = ((v,w)) + (q,r), \quad \|(v,q)\|_{H_h}^2 = \|v\|^2 + |q|^2,$$

and K_h be a non-void, convex and compact subset of H_h defined by

$$K_h = \left\{(v,q) \in H_h : \quad \|v\| \le \frac{\gamma_0}{\nu}|f|, \quad |q| \le \frac{c_0\gamma_0^3}{\alpha\nu^2}|f|^2 + \frac{\gamma_0}{\alpha}|f|\right\}.$$

We now define a continuous mapping from K_h into H_h as follows: Given $(\bar{v},\bar{q}) \in K_h$ find $(v,q) = \Psi(\bar{v},\bar{q})$ such that for all $(w,r) \in H_h$

$$\mathcal{B}_h((v,q);(w,r)) + b(\bar{v},v,w) = (f,w). \tag{3.14}$$

Taking $(w,r) = (v,q)$ in (3.14) and using (2.2–2.5) and (3.11–3.12), we obtain

$$\nu\|v\|^2 \le \gamma_0|f|\|v\|,$$
$$\alpha(\|v\| + |q|) \le \gamma_0|f| + c_0\gamma_0\|\bar{v}\|\|v\| \le \gamma_0|f| + c_0\nu^{-2}\gamma_0^3|f|^2,$$

which implies $\Psi(\bar{v},\bar{q}) = (v,q) \in K_h$. By the fixed-point theorem (see [1]), the mapping $\Psi(\bar{v},\bar{q})$ has at least a fixed point $(u_h,p_h) \in K_h$, namely, $(u_h,p_h) \in K_h$ is a stabilized finite-element solution of problem (3.5).

Next, we shall prove that problem (3.5) has only one solution (u_h,p_h). In fact, if (v_h,q_h) also satisfies formulation (3.5), then for all $(w,r) \in (X_h,M_h)$

$$\mathcal{B}_h((u_h - v_h, p_h - q_h);(w,r)) = b(v_h - u_h, u_h, w) + b(v_h, v_h - u_h, w). \tag{3.15}$$

Taking $(w,r) = (u_h - v_h, p_h - q_h)$ in (3.15) and using (2.2) and (2.4–2.5), we have

$$\nu\|u_h - v_h\|^2 \le c_0\gamma_0\|u_h\|\|u_h - v_h\|^2 \le c_0\frac{\gamma_0^2}{\nu}|f|\|u_h - v_h\|^2,$$

which together with the fact

$$\nu - c_0\frac{\gamma_0^2}{\nu}|f| = \nu\left(1 - c_0\frac{\gamma_0^2}{\nu^2}|f|\right) > 0,$$

gives $u_h = v_h$. Using again (3.15), (3.11) and (2.5), we obtain $\alpha|p_h - q_h| \le 0$, namely $p_h = q_h$. □

4. Error estimates

In order to derive error estimates of the stabilized finite-element solution (u_h,p_h), we also need the Galerkin projection $(R_h,Q_h):(X,M)\to(X_h,M_h)$ defined by

$$\mathcal{B}_h((R_h(v,q) - v, Q_h(v,q) - q);(v_h,q_h)) = 0, \quad \forall (v_h,q_h) \in (X_h,M_h), \tag{4.1}$$

for each $(v,q) \in (X,M)$. Note that, due to Theorem 3.3, (R_h,Q_h) is well defined. By using an exact similar argument to the one used by Layton and Tobiska in [18], we may obtain the following approximation properties.

Lemma 4.1. *Under the assumptions of Theorem 3.3, the projection* (R_h, Q_h) *satisfies*

$$|v - R_h(v,q)| + h\|v - R_h(v,q)\| + h|q - Q_h(v,q)| \le c_5 h(\|v\| + |q|), \tag{4.2}$$

for all $(v,q) \in (X, M)$ *and*

$$|v - R_h(v,q)| + h\|v - R_h(v,q)\| + h|q - Q_h(v,q)| \le c_5 h^2(|Av| + \|q\|_1), \tag{4.3}$$

for all $(v,q) \in (D(A), H^1(\Omega) \cap M)$.

Proof. The stability of the projection follows simply by Theorem 3.3, namely

$$\begin{aligned}\|R_h(v,q)\| + |Q_h(v,q)| &\le \alpha^{-1} \sup_{(v_h,q_h)\in(X_h,M_h)} \frac{\mathcal{B}_h((v,q);(v_h,q_h))}{\|v_h\| + |q_h|} \\ &\le \alpha^{-1}\gamma_3(\|v\| + |q|), \quad \forall (v,q) \in (X, M).\end{aligned} \tag{4.4}$$

Now the triangle inequality gives

$$\|v - R_h(v,q)\| + |q - Q_h(v,q)| \le (1 + \alpha^{-1}\gamma_3)(\|v\| + |q|), \quad \forall (v,q) \in (X, M). \tag{4.5}$$

Next, we introduce the dual problem: find $(\Psi, \Phi) \in (X, M)$ such that

$$\mathcal{B}_h((\Psi, \Phi); (w, r)) = (v - R_h(v,q), w), \quad \forall (w, r) \in (X, M).$$

Now, setting $w = v - R_h(v,q)$, $r = q - Q_h(v,q)$ and using the projection property, the regularity assumption **(A1)** and (3.6–3.7), we have, for $(\Psi_h, \Phi_h) = (I_h\Psi, J_h\Phi) \in (X_h, M_h)$,

$$\begin{aligned}|v - R_h(v,q)|^2 &= \mathcal{B}_h((\Psi - \Psi_h, \Phi - \Phi_h); (v - R_h(v,q), q - Q_h(v,q))) \\ &\le C(\|\Psi - \Psi_h\| + |\Phi - \Phi_h|)(\|v - R_h(v,q)\| + |q - Q_h(v,q)|) \\ &\le Ch(\|v - R_h(v,q)\| + |q - Q_h(v,q))(\|\Psi\|_2 + \|\Phi\|_1) \\ &\le Ch(\|v - R_h(v,q)\| + |q - Q_h(v,q)|)|v - R_h(v,q)|,\end{aligned} \tag{4.6}$$

where $C > 0$ is a general constant depending on the data (Ω, ν, β). Combining (4.5) and (4.6) gives (4.2).

Let $(v,q) \in (D(A), H^1(\Omega) \cap M)$. Then, using the standard interpolation $(I_h v, J_h p) \in (X_h, M_h)$, and Theorem 3.3, we have

$$\|I_h v - R_h(v,q)\| + |J_h q - Q_h(v,q)| \ \le\ \alpha^{-1} \sup_{(v_h,q_h)\in(X_h,M_h)} \frac{\mathcal{B}_h((I_h v - v, J_h q - q); (v_h, q_h))}{\|v_h\| + |q_h|}.$$

Thus the triangle inequality and approximate properties (3.9–3.10) give

$$\|v - R_h(v,q)\| + |q - Q_h(v,q)| \le Ch(|Av| + \|q\|_1). \tag{4.7}$$

It now follows from (4.6) and (4.7) that

$$|v - R_h(v,q)| \le Ch^2(|Av| + \|q\|_1). \tag{4.8}$$

Thus, (4.7) and (4.8) imply (4.3). □

Next, we will derive the following error estimates of the finite-element solution (u_h, p_h) defined in Section 3.

Theorem 4.2. *Assume that the assumptions of Theorem* 2.1 *and Theorem* 3.3 *hold. Then the stabilized finite-element solution* (u_h, p_h) *satisfies the error estimates:*

$$|u-u_h|+h(\|u-u_h\|+|p-p_h|)\leq ch^2, \tag{4.9}$$

where $c>0$ *is a general constant depending on the data* $(\Omega, \nu, \beta_0, f)$.

Proof. Since $\mathcal{C}_h(p, q_h)=0, \forall p\in H^1(\Omega)\cap M, q_h\in M_h$, we derive from (2.7) and (3.5) that for all $(v,q)\in(X_h, M_h)$

$$\mathcal{B}_h((e_h, \eta_h);(v,q))+b(u-R_h(u,p)+e_h, u, v)+b(u_h, u-R_h(u,p)+e_h, v)=0, \tag{4.10}$$

where $e_h=R_h(u,p)-u_h$ and $\eta_h=Q_h(u,p)-p_h$. Taking $(v,q)=(e_h,\eta_h)$ in (4.10) and using (2.4), we arrive at

$$\begin{aligned}&\nu\|e_h\|^2+\beta_0\mathcal{C}_h(\eta_h,\eta_h)+b(e_h,u,e_h)\\&\quad\leq|b(u-R_h(u,p),u,e_h)|+|b(u_h,u-R_h(u,p),e_h)|.\end{aligned} \tag{4.11}$$

We find from (2.5), (2.9), (3.13) and (4.3) that

$$\nu\|e_h\|^2-|b(e_h,u,e_h)|\geq\nu\|e_h\|^2-c_0\gamma_0\|u\|\|e_h\|^2\geq\nu\left(1-c_0\gamma_0^2|f|\nu^{-2}\right)\|e_h\|^2, \tag{4.12}$$

$$\begin{aligned}&|b(u_h,u-R_h(u,p),e_h)|+|b(u-R_h(u,p),u,e_h)|\\&\quad\leq c_0\gamma_0(\|u\|+\|u_h\|)\|e_h\|\|u-R_h(u,p)\|\leq ch\|e_h\|.\end{aligned} \tag{4.13}$$

Combining (4.11) with (4.12–4.13) yields

$$\|e_h\|\leq ch. \tag{4.14}$$

Moreover, by using (2.5–2.6), (2.9), (4.3) and (4.14), we have

$$\begin{aligned}&|b(u_h,u-R_h(u,p),e_h)|+|b(u-R_h(u,p),u,e_h)|\\&\quad\leq|b(u,u-R_h(u,p),e_h)|+|b(u-R_h(u,p),u,e_h)|\\&\qquad+|b(u-R_h(u,p),u-R_h(u,p),e_h)|+|b(e_h,u-R_h(u,p),e_h)|\\&\quad\leq c_1|Au||u-R_h(u,p)|\|e_h\|\\&\qquad+c_0\gamma_0(\|u-R_h(u,p)\|+\|e_h\|)\|u-R_h(u,p)\|\|e_h\|\leq ch^2\|e_h\|.\end{aligned} \tag{4.15}$$

Combining (4.11–4.12) with (4.15) gives

$$\|e_h\|\leq ch^2, \tag{4.16}$$

Moreover, one finds from (4.3), (4.16) and (2.9) that

$$|u-u_h|\leq|e_h|+|u-R_h(u,p)|\leq\gamma_0\|e_h\|+c_5h^2(|Au|+\|p\|_1)\leq ch^2, \tag{4.17}$$

$$\|u-u_h\|\leq\|e_h\|+\|u-R_h(u,p)\|\leq ch^2+c_5h(|Au|+\|p\|_1)\leq ch. \tag{4.18}$$

Using again (3.11), (4.10), (2.9) and (3.13), we obtain

$$|\eta_h|\leq\alpha^{-1}c(\|u\|+\|u_h\|)\|u-u_h\|\leq c\|u-u_h\|. \tag{4.19}$$

It follows from (4.3), (4.18–4.19) and (2.9) that

$$|p-p_h|\leq|p-Q_h(u,p)|+|\eta_h|\leq ch(|Au|+\|p\|_1)+c\|u-u_h\|\leq ch. \tag{4.20}$$

Combining (4.17–4.18) with (4.20) yields (4.9). □

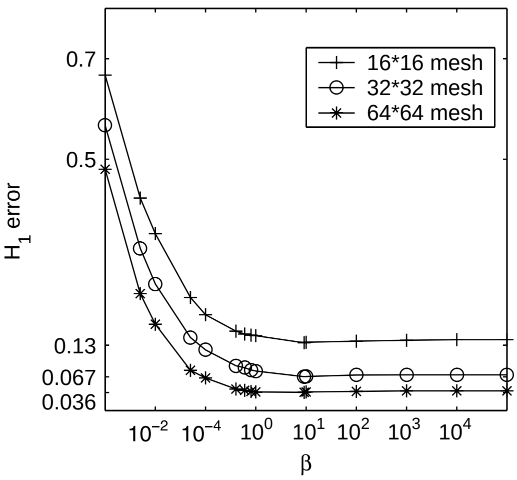

Figure 3. The relative velocity error curve with respect to β.

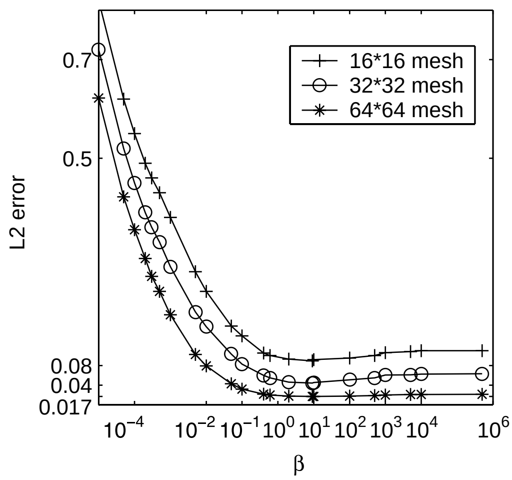

Figure 4. The relative pressure error curve with respect to β.

Table 1. Numerical results of the stabilized finite-element method.

h	CPU(s)	$\frac{\|u-u_h\|}{\|u\|}$	$\frac{\lvert u-u_h\rvert}{\lvert u\rvert}$	$\frac{\lvert p-p_h\rvert}{\lvert p\rvert}$
1/16	2	0·1349377549	0·05282056	0·0890762051
1/32	56	0·0674981988	0·04425722	0·0442804064
1/64	440	0·036532981	0·028694634	0·0170028269

5. Numerical Examples

In this section we assess the performance of the stabilized finite-element method described in Section 3. We consider a unit-square domain with a driven-cavity flow solution, which is a very popular problem in testing various numerical method. The graphs of the flow obtained by finite-element method are documented in [19] and [20]. In this paper, we set that the exact solution is given by

$$u(x,y)=(u_1(x,y),u_2(x,y)),\quad p(x,y)=10(2x-1)(2y-1),$$
$$u_1(x,y)=10x^2(x-1)^2y(y-1)(2y-1),\quad u_2(x,y)=-10x(x-1)(2x-1)y^2(y-1)^2,$$

with $\nu=0.005$ and f is determined by (2.1).

The first issue to be considered here is the stabilized finite-element method being stable with respect to the stabilization parameter β. We use the Q_1-P_0 quadrilateral element with $h=\frac{1}{16}$, $h=\frac{1}{32}$, $h=\frac{1}{64}$ and plot the relative velocity error $e_h=\frac{\|u-u_h\|}{\|u\|}$ and pressure error $\eta_h=\frac{|p-p_h|}{|p|}$ as β changing in Figures 3 and 4, respectively.

Next, we provide the convergence accuracy of the stabilized finite-element method with $h=\frac{1}{16}$ and $h=\frac{1}{32}$ and $h=\frac{1}{64}$ when the best parameter value $\beta=9{\cdot}18$ is used to solve the flow problem on a uniformly refined sequence of grids in Table 1. This method clearly shows the anticipated first-order convergence rate as $h\to 0$.

A practical problem description is shown in Figures 5 and 6. The results are presented graphically in Figures 7–10 for $\beta=10^{-4}$, $\beta=0{\cdot}1$, $\beta=9{\cdot}18$ and $\beta=10{,}000$, respectively.

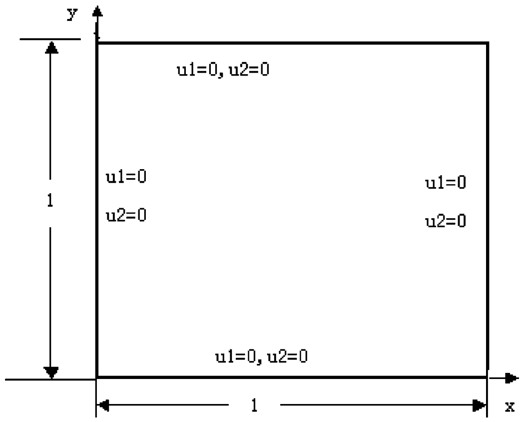

Figure 5. Driven cavity flow.

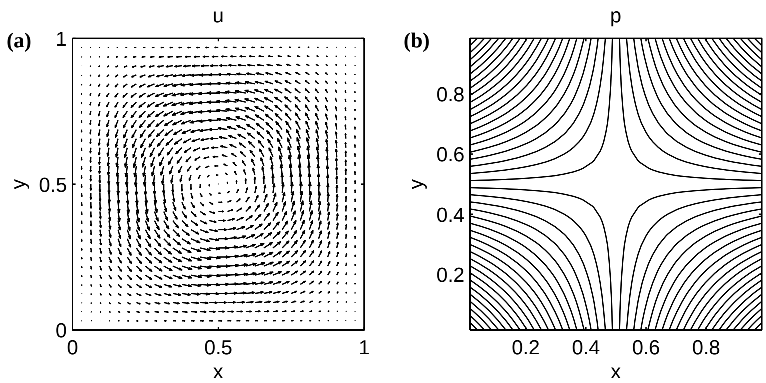

Figure 6. Driven cavity flow: (a) velocity vectors and (b) pressure contours.

The Figures 7–10 show that there exists a threshold for β such that the results are good as $\beta \geq 9{\cdot}18$, but the results are extremely wrong as $\beta \to 0$.

6. Conclusions

In this paper, we have provided a theoretical analysis of the stabilized finite-element method for the two-dimensional stationary Navier-Stokes equations. The analysis is a extension of the work of Braess [9], Kechkar and Silvester [4,5], Kay and Silvester [6], Norburn and Silvester [7] and Pitkäranta and Saarinen [10] for solving the stationary Stokes equations. The discretization is based on the finite-element space pair (X_h, M_h) for the approximation of the velocity and the pressure, constructed by using the $Q_1 - P_0$ quadrilateral element or the $P_1 - P_0$ triangular element and some local stabilized bilinear form $\beta C_h(p_h, q)$ established on the macroelements which satisfy the macroelement connectivity condition; while the above finite-element space pair (X_h, M_h) without introducing the above stabilized bilinear form can not be used to solve the two-dimensional stationary Navier-Stokes equations.

For the stabilized finite-element solution (u_h, p_h) we have established the existence, stability and the optimal error estimate. Finally, we have presented some numerical tests which show that the stabilized finite-element method is stable and efficient numerically for solving

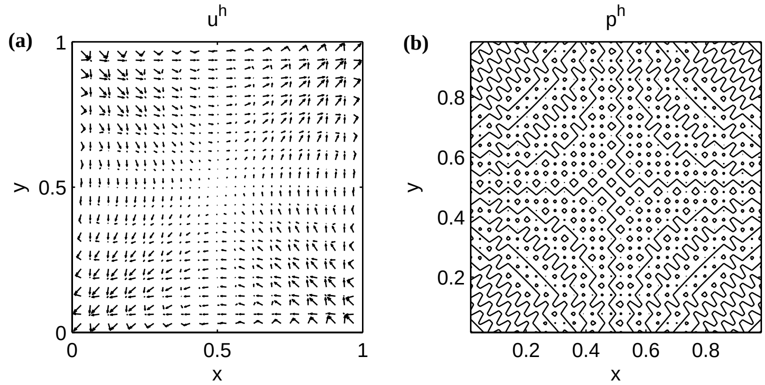

Figure 7. Driven cavity flow $\beta = 10^{-4}$: (a) velocity vectors and (b) pressure contours.

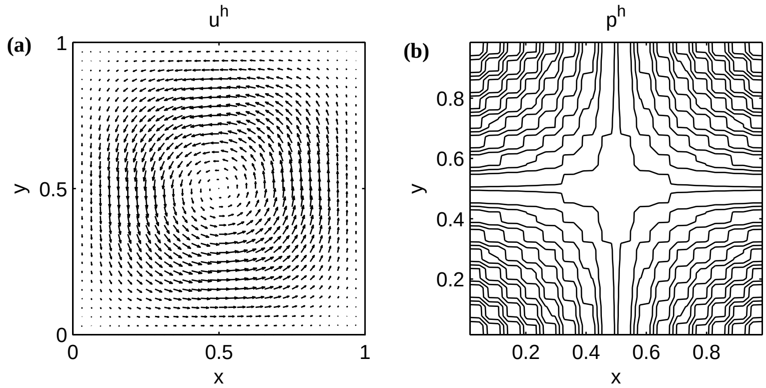

Figure 8. Driven cavity flow $\beta = 0.1$: (a) velocity vectors and (b) pressure contours.

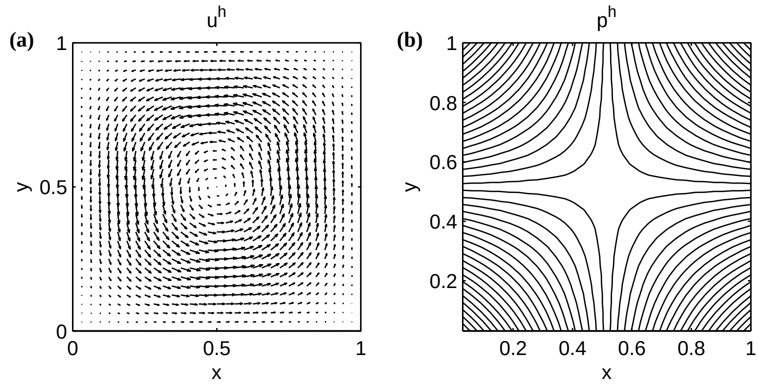

Figure 9. Driven cavity flow $\beta = 9{\cdot}18$: (a) velocity vectors and (b) pressure contours.

the two-dimensional stationary Navier-Stokes equations; therefore, it is suitable to solve practical engineering problems arising in fluid dynamics. Furthermore, the method helps to solve non-stationary two-dimensional or three-dimensional Navier-Stokes equations which will be discussed in our further work.

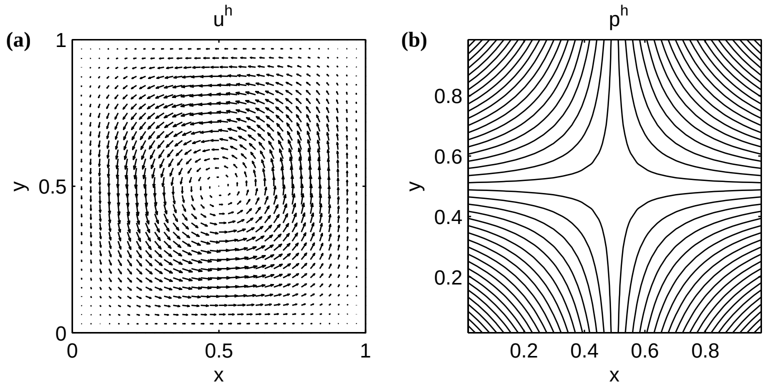

Figure 10. Driven cavity flow $\beta = 10{,}000$: (a) velocity vectors and (b) pressure contours.

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References

1. V. Girault and P.A. Raviart, *Finite Element Method for Navier-Stokes Equations: Theory and Algorithms*. Berlin, Heidelberg: Springer-Verlag (1981) 202 pp.
2. J. Boland and R.A. Nicolaides, Stability of finite elements under divergence constraints. *SIAM J. Numer. Anal*. 20 (1983) 722–731.
3. R. Stenberg, Analysis of mixed finite elements for the Stokes problem: A unified approach. *Math. Comp.* 42 (1984) 9–23.
4. N. Kechkar and D. Silvester, Analysis of locally stabilized mixed finite element methods for the Stokes problem. *Math. Comp.* 58 (1992) 1–10.
5. D.J. Silvester and N. Kechkar, Stabilised bilinear-constant velocity–pressure finite elements for the conjugate gradient solution of the Stokes problem. *Comp. Methods Appl. Mech. Engng.* 79 (1990) 71–86.
6. D. Kay and D. Silvester, *A posteriori* error estimation for stabilized mixed approximations of the Stokes equations. *SIAM J. Sci. Comp.* 21 (2000) 1321–1337.
7. S. Norburn and D. Silvester, Stabilised vs. stable mixed methods for incompressible flow. *Comput. Methods Appl. Mech. Engng.* 166 (1998) 1–10.
8. D. Silvester and A. Wathen, Fast interative solution of stabilised Stokes systems, Part II: Using general block preconditioners. *SIAM J. Numer. Anal.* 31 (1994) 1352–1367.
9. D. Braess, *Finite Elements, Theory, Fast Solvers and Applications in Solid Mechnics.* Cambridge: Cambridge University Press (1997) 323 pp.
10. J. Pitkäranta and T. Saarinen, A multigrid version of a simple finite element method for the Stokes problem. *Math. Comput.* 45 (1985) 1–14.
11. J.G. Heywood and R. Rannacher, Finite element approximation of the nonstationary Navier-Stokes problem, I: Regularity of solutions and second-order error estimates for spatial discretization. *SIAM J. Numer. Anal.* 19 (1982) 275–311.
12. R.B. Kellogg and J.E. Osborn, A regularity result for the Stokes problem in a convex polygon. *J. Funct. Anal.* 21 (1976) 397–431.
13. A. Ait Ou Ammi and M. Marion, Nonlinear Galerkin methods and mixed finite elements: Two-grid algorithms for the Navier-Stokes equations. *Numer. Math.* 68 (1994) 189–213.
14. R. Temam, *Navier-Stokes Equations, Theory and Numerical Analysis*. Amsterdam: North-Holland (1984) 526 pp.
15. P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*. Amsterdam: North-Holland (1978) 519 pp.

16. R.L. Sani, P.M. Gresho, R.L. Lee and D.F. Griffiths, The cause and cure(?) of the spuious pressures generated by certain finite element method solutions of the incompressible Navier-Stokes equations. Parts 1 and 2. *Int. J. Numer. Methods Fluids* 1 (1981) 17–43; 171–206.
17. T.J.R. Hughes and L.P. Franca, A new finite element formulation for CFD: VII. The Stokes problem with various well-posed boundary conditions: Symmetric formulations that converge for all velocity/pressure spaces. *Comput. Methods Appl. Mech. Engng.* 65 (1987) 85–97.
18. W. Layton and L. Tobiska, A two-level method with backtraking for the Navier-Stokes equations. *SIAM J. Numer. Anal.* 35 (1998) 2035–2054.
19. A.J. Baker, *Finite Element Computational Fluid Mechanics*, Washington: Hemisphere Pub. Corp. (1983) 510 pp.
20. T.J.R. Hughes, W.K. Liu and A. Brooks, Review of finite element analysis of incompressible viscous flow by the penalty function formulations. *J. Comp. Phys.* 30 (1979) 1–60.